

Foundations of AI and Machine Learning

Constraint Satisfaction Problems: Introduction and Examples

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Introduction

Constraints

What is a Constraint?

a condition that every solution to a problem must satisfy

Examples: Where do constraints occur?

- **mathematics**: requirements on solutions of optimization problems (e.g., equations, inequalities)
- **software testing**: specification of invariants to check data consistency (e.g., assertions)

Constraint Satisfaction Problems: Informally

Given:

- set of **variables** with corresponding domains
- set of **constraints** that the variables must satisfy
 - most commonly **binary**, i.e., every constraint refers to **two** variables

Solution:

- **assignment** to the variables that satisfies all constraints

Examples

Examples

Examples

- 8 queens problem
- Sudoku
- graph coloring
- satisfiability in propositional logic

more complex examples:

- systems of equations and inequalities
- database queries

Example: 8 Queens Problem

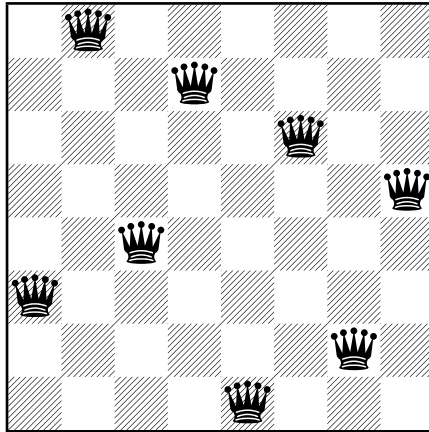
8 Queens Problem

How can we

- place **8 queens** on a chess board
 - such that **no two queens threaten each other?**
-
- originally proposed in 1848
 - **variants**: board size; other pieces; higher dimension

There are **92 solutions**, or **12 solutions** if we do not count symmetric solutions (under rotation or reflection) as distinct.

8 Queens Problem: Example Solution



example solution for the 8 queens problem

Example: Sudoku

Sudoku

How can we

- completely fill an already partially filled 9×9 matrix with numbers between 1–9
- such that each row, each column, and each of the nine 3×3 blocks contains every number exactly once?

2	5			3		9		1
	1				4			
4		7				2		8
		5	2					
				9	8	1		
	4				3			
			3	6			7	2
	7							3
9		3				6		4

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6	1	9	8	2	4	3	5	7
4	3	7	9	1	5	2	6	8
3	9	5	2	7	1	4	8	6
7	6	2	4	9	8	1	3	5
8	4	1	6	5	3	7	2	9
1	8	4	3	6	9	5	7	2
5	7	6	1	4	2	8	9	3
9	2	3	5	8	7	6	1	4

Sudoku: Small Example

variables:

$$V_{xy}, V_{ni} \in \{1, 2, 3, 4\}$$

$$V_{22} \in \{1, 2, 3, 4\}, \dots$$

constraints:

$$\text{box: } V_{11} \neq V_{21}$$

$$V_{11} \neq V_{22}$$

$$V_{11} \neq V_{12}$$

⋮

$$V_{21} \neq V_{22}$$

$$V_{12} \neq V_{21}$$

$$V_{12} \neq V_{22}$$

row:

$$V_{11} \neq V_{12}$$

$$V_{11} \neq V_{13}$$

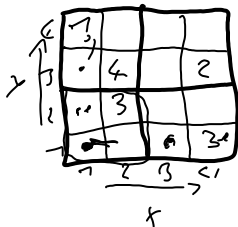
⋮

columns:

$$V_{11} \neq V_{12}$$

$$V_{11} \neq V_{13}$$

⋮



Sudoku: Trivia

- well-formed Sudokus have **exactly one** solution
- to achieve well-formedness, ≥ 17 cells must be filled already (McGuire et al., 2012)
- 6 670 903 752 021 072 936 960 solutions
- only 5 472 730 538 “non-symmetrical” solutions

Example: Graph Coloring

Graph Coloring

How can we

- color the vertices of a given graph using k colors
- such that two neighboring vertices never have the same color?

(The graph and k are problem parameters.)

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- easy for $k = 2$ (also for general graphs)

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Relationship to Sudoku?

Four Color Problem

famous problem in mathematics: **Four Color Problem**

- Is it always possible to color a **planar** graph with 4 colors?
- conjectured by Francis Guthrie (1852)
- 1890 first proof that 5 colors suffice
- several wrong proofs surviving for over 10 years

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- 1890 first proof that 5 colors suffice
- several wrong proofs surviving for over 10 years
- solved by Appel and Haken in 1976: 4 colors suffice
- Appel and Haken reduced the problem to 1936 cases, which were then checked by computers
- first famous mathematical problem solved (partially) by computers
 - ↪ led to controversy: is this a mathematical proof?

Numberphile video:

<https://www.youtube.com/watch?v=NgBK43jB4rQ>

Satisfiability in Propositional Logic

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How can we

- assign **truth values** (true/false) to a set of propositional variables
- such that a given set of **clauses** (formulas of the form $X \vee \neg Y \vee Z$) is satisfied (true)?

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remarks:

- NP-complete (Cook 1971; Levin 1973)
- formulas expressed as clauses (instead of arbitrary propositional formulas) is no restriction
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relationship to previous problems (e.g., Sudoku)?

Practical Applications

- There are **thousands** of practical applications of constraint satisfaction problems.
- This statement is true already for the satisfiability problem of propositional logic.

some examples:

- verification of hardware and software
- timetabling (e.g., generating time schedules, room assignments for university courses)
- assignment of frequency spectra (e.g., broadcasting, mobile phones)

Summary

Summary

- **constraint satisfaction:**
 - find **assignment** for a set of **variables**
 - with given **variable domains**
 - that satisfies a given set of **constraints**.
- **examples:**
 - 8 queens problem
 - Sudoku
 - graph coloring
 - satisfiability in propositional logic
 - many practical applications