Basics of AI and Machine Learning

Constraint Satisfaction Problems: Constraint Networks

Daniel Gnad

Linköping University

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Constraint Networks

Constraint Networks: Informally

Constraint Networks: Informal Definition

A constraint network is defined by

- a finite set of variables
- a finite domain for each variable
- a set of constraints (here: binary relations)

The objective is to find a solution for the constraint network, i.e., an assignment of the variables that complies with all constraints.

Informally, people often just speak of constraint satisfaction problems (CSP) instead of constraint networks.

More formally, a "CSP" is the algorithmic problem of finding a solution for a constraint network.

Constraint Networks: Formally

Definition (binary constraint network)

A (binary) constraint network
is a 3-tuple $\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$ such that:

- V is a non-empty and finite set of variables,
- dom is a function that assigns a non-empty and finite domain to each variable $v \in V$, and
- $(R_{uv})_{u,v \in V, u \neq v}$ is a family of binary relations (constraints) over V where for all $u \neq v$: $R_{uv} \subseteq \text{dom}(u) \times \text{dom}(v)$

possible generalizations:

- infinite domains (e.g., $\text{dom}(v) = \mathbb{Z}$)
- constraints of higher arity
  (e.g., satisfiability in propositional logic)

Binary Constraints

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- For variables u, v, the constraint R_{uv} expresses which joint assignments to u and v are allowed in a solution.
- If $R_{uv} = \mathrm{dom}(u) \times \mathrm{dom}(v)$, the constraint is trivial: there is no restriction, and the constraint is typically not given explicitly in the constraint network description (although it formally always exists!).
- Constraints R_{uv} and R_{vu} refer to the same variables. Hence, usually only one of them is given in the description.

Unary Constraints

unary constraints:

- It is often useful to have additional restrictions on single variables as constraints.
- Such constraints are called unary constraints.
- A unary constraint R_v for $v \in V$ corresponds to a restriction of $\mathrm{dom}(v)$ to the values allowed by R_v.
- Formally, unary constraints are not necessary, but they often allow us to describe constraint networks more clearly.

Compact Encodings and General Constraint Solvers

Constraint networks allow for compact encodings of large sets of assignments:

- Consider a network with n variables with domains of size k.

$\rightsquigarrow$ k^n assignments

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$\rightsquigarrow$ compact encoding size

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⇝ compact encoding size

- We observe: The number of assignments is exponentially larger than the description of the constraint network.
- As a consequence, such descriptions can be used as inputs of general constraint solvers.

Example

Example: Sudoku

Sudoku as Constraint Network

- variables: $V = \{v_{ij} \mid 1 \leq i, j \leq 9\}$; v_{ij}: Value row i, column j
- domains: $\text{dom}(v) = \{1, \dots, 9\}$ for all $v \in V$
- unary constraints: $R_{v_{ij}} = \{k\}$,
  if $\langle i, j \rangle$ is a cell with predefined value k
- binary constraints: for all $v_{ij}, v_{i'j'} \in V$, we set
  $R_{v_{ij} v_{i'j'}} = \{\langle a, b \rangle \in \{1, \dots, 9\} \times \{1, \dots, 9\} \mid a \neq b\}$,
  if $i = i'$ (same row), or $j = j'$ (same column),
  or $\langle \lceil \frac{i}{3} \rceil, \lceil \frac{j}{3} \rceil \rangle = \langle \lceil \frac{i'}{3} \rceil, \lceil \frac{j'}{3} \rceil \rangle$ (same block)

2	5			3		9		1
	1				4			
4		7				2		8
		5	2					
				9	8	1		
	4				3			
			3	6			7	2
	7							3
9		3				6		4

Assignments and Consistency

Assignments

Definition (assignment, partial assignment)

Let $\mathcal{C} = \langle V, \mathrm{dom}, (R_{uv}) \rangle$ be a constraint network.
A partial assignment of $\mathcal{C}$ (or of V) is a function

$$\alpha : V' \to \bigcup_{v \in V} \mathrm{dom}(v)$$

with $V' \subseteq V$ and $\alpha(v) \in \mathrm{dom}(v)$ for all $v \in V'$.

If $V' = V$, then α is also called total assignment (or assignment).

- ⇝ partial assignments assign values to some or to all variables
- ⇝ (total) assignments are defined on all variables

Consistency

Definition (inconsistent, consistent, violated)

A partial assignment α of a constraint network $\mathcal{C}$ is called inconsistent if there are variables u, v such that α is defined for both u and v, and $\langle \alpha(u), \alpha(v) \rangle \notin R_{uv}$.

In this case, we say α violates the constraint R_{uv}.

A partial assignment is called consistent if it is not inconsistent.

trivial example: The empty assignment is always consistent.

Solution

Definition (solution, solvable)

Let $\mathcal{C}$ be a constraint network.

A consistent and total assignment of $\mathcal{C}$ is called a solution of $\mathcal{C}$.

If a solution of $\mathcal{C}$ exists, $\mathcal{C}$ is called solvable.

If no solution exists, $\mathcal{C}$ is called inconsistent.

Consistency vs. Solvability

Note: Consistent partial assignments α cannot necessarily be extended to a solution.

It only means that so far (i.e., on the variables where α is defined) no constraint is violated.

Example (4 queens problem): $\alpha = \{v_1 \mapsto 1, v_2 \mapsto 4, v_3 \mapsto 2\}$

	v_1	v_2	v_3	v_4
1	q			
2			q	
3				
4		q		

Tightness of Constraint Networks

Definition (tighter, strictly tighter)

Let $\mathcal{C} = \langle V, \mathrm{dom}, R_{uv} \rangle$ and $\mathcal{C}' = \langle V, \mathrm{dom}', R'_{uv} \rangle$ be constraint networks with equal variable sets V.

$\mathcal{C}$ is called tighter than $\mathcal{C}'$, in symbols $\mathcal{C} \sqsubseteq \mathcal{C}'$, if

- $\mathrm{dom}(v) \subseteq \mathrm{dom}'(v)$ for all $v \in V$
- $R_{uv} \subseteq R'_{uv}$ for all $u, v \in V$ (including trivial constraints).

If at least one of these subset equations is strict, then $\mathcal{C}$ is called strictly tighter than $\mathcal{C}'$, in symbols $\mathcal{C} \sqsubset \mathcal{C}'$.

Equivalence of Constraint Networks

Definition (equivalent)

Let $\mathcal{C}$ and $\mathcal{C}'$ be constraint networks with equal variable sets.

$\mathcal{C}$ and $\mathcal{C}'$ are called equivalent, in symbols $\mathcal{C} \equiv \mathcal{C}'$,
if they have the same solutions.

Outline and Summary

CSP Algorithms

In the following parts, we will consider solution algorithms for constraint networks.

basic concepts:

- search: check partial assignments systematically
- backtracking: discard inconsistent partial assignments
- inference: derive equivalent, but tighter constraints to reduce the size of the search space

Summary

- formal definition of constraint networks:
  variables, domains, constraints
- compact encodings of exponentially many configurations
- unary and binary constraints
- assignments: partial and total
- consistency of assignments; solutions
- tightness of constraints
- equivalence of constraints