# Foundations of AI and Machine Learning

## Propositional Logic: Modeling Example

Daniel Gnad

Linköping University

# Modeling Example

# Modeling in Propositional Logic

- Propositional Logic is a low-level language.
- Can be intricate to model non-trivial applications.
- Solvers are extremely powerful and solve huge problems efficiently.

# Modeling: Sudoku

| | | | | |
|---|---|---|---|---|
| 4 | 1 | | | |
| 3 | | 4 | | 2 |
| 2 | | 3 | | |
| 1 | | | 2 | |
| | 1 | 2 | 3 | 4 |

# Modeling: Sudoku

Propositions:

- Assignment of number to every grid cell

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  e.g. $c_{12}v_3$: cell (1,2) has value 3.

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- Assignment of number to every grid cell
  e.g. $c_{12}v_3$: cell (1,2) has value 3.
- $\Sigma = \{c_{11}v_1, c_{11}v_2, c_{11}v_3, c_{11}v_4, c_{12}v_1, c_{12}v_2, c_{12}v_3, c_{12}v_4, \ldots\}$

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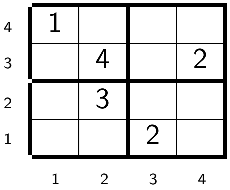
- 64 propositions in total.



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## Any issues?

- Nothing prevents all propositions $c_{11}v_x$ to be true.

$\rightsquigarrow$ Make sure that exactly one number is assigned to every cell.

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## Any issues?

- Nothing prevents all propositions $c_{11}v_x$ to be true.

$\rightsquigarrow$ Make sure that exactly one number is assigned to every cell.

$$(c_{12}v_1 \land \neg c_{12}v_2 \land \neg c_{12}v_3 \land \neg c_{12}v_4) \lor$$
$$(\neg c_{12}v_1 \land c_{12}v_2 \land \neg c_{12}v_3 \land \neg c_{12}v_4) \lor$$
$$(\neg c_{12}v_1 \land \neg c_{12}v_2 \land c_{12}v_3 \land \neg c_{12}v_4) \lor$$
$$(\neg c_{12}v_1 \land \neg c_{12}v_2 \land \neg c_{12}v_3 \land c_{12}v_4)$$

# Model Sudoku Rules I

- Every cell in box has different number:
  $\neg(c_{11}v_1 \wedge c_{12}v_1) \wedge \neg(c_{11}v_2 \wedge c_{12}v_2) \wedge \neg(c_{11}v_3 \wedge c_{12}v_3) \wedge \neg(c_{11}v_4 \wedge c_{12}v_4)$

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  $\neg(c_{11}v_3 \wedge c_{12}v_3) \wedge \neg(c_{11}v_4 \wedge c_{12}v_4)$

  Similar for all pairs of cells in a box.
  Analogous formulas for all 4 boxes.

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- Every cell in row has different number:
Same as above, but for all pairs of cells in each row.

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  $\neg(c_{11}v_1 \wedge c_{12}v_1) \wedge \neg(c_{11}v_2 \wedge c_{12}v_2) \wedge \neg(c_{11}v_3 \wedge c_{12}v_3) \wedge \neg(c_{11}v_4 \wedge c_{12}v_4)$

  Similar for all pairs of cells in a box.
  Analogous formulas for all 4 boxes.

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- Every cell in row has different number:
  Same as above, but for all pairs of cells in each row.
- Every cell in column has different number.
  Same as above, but for all pairs of cells in each column.

# Model Sudoku – “Full” Model

**Propositions:**

$\Sigma = \{c_{11}v_1, c_{11}v_2, c_{11}v_3, c_{11}v_4, c_{12}v_1, c_{12}v_2, \ldots\}$
(64 propositions in total)

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**Consistency:**

$(c_{12}v_1 \wedge \neg c_{12}v_2 \wedge \neg c_{12}v_3 \wedge \neg c_{12}v_4) \vee$
$(\neg c_{12}v_1 \wedge c_{12}v_2 \wedge \neg c_{12}v_3 \wedge \neg c_{12}v_4) \vee \ldots$

**Sudoku rules:**

For every pair of cells $c_{ab}, c_{cd}$ in every box/row/column:
$\neg(c_{11}v_1 \wedge c_{12}v_1) \wedge \neg(c_{11}v_2 \wedge c_{12}v_2) \wedge$
$\neg(c_{11}v_3 \wedge c_{12}v_3) \wedge \neg(c_{11}v_4 \wedge c_{12}v_4)$

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$\Sigma = \{c_{11}v_1, c_{11}v_2, c_{11}v_3, c_{11}v_4, c_{12}v_1, c_{12}v_2, \ldots\}$
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**Sudoku rules:**

For every pair of cells $c_{ab}, c_{cd}$ in every box/row/column:
$\neg(c_{11}v_1 \wedge c_{12}v_1) \wedge \neg(c_{11}v_2 \wedge c_{12}v_2) \wedge$
$\neg(c_{11}v_3 \wedge c_{12}v_3) \wedge \neg(c_{11}v_4 \wedge c_{12}v_4)$

Construct one large formula as conjunction of these formulas.

# Summary

# Summary

- Many problems can be modeled in propositional logic.
- Can be challenging to formalize applications.
- Solvers can solve huge instances efficiently.
- Enumerate many (all) possible solutions.
- Guarantees!