Basics of AI and Machine Learning

Propositional Logic: Basics

Daniel Gnad

Linköping University

Slides modified from Basel AI group, with permission

Classification

classification:

Propositional Logic

environment:

- static vs. dynamic
- deterministic vs. non-deterministic vs. stochastic
- fully vs. partially vs. not observable
- discrete vs. continuous
- single-agent vs. multi-agent

problem solving method:

- problem-specific vs. general vs. learning

(applications also in more complex environments)

Motivation

Propositional Logic: Motivation

propositional logic

- modeling and representing problems and knowledge
- basics for general problem descriptions and solving strategies ($\rightsquigarrow$ automated planning)
- allows for automated reasoning

Relationship to CSPs

- satisfiability problem in propositional logic can be viewed as non-binary CSP over $\{\mathbf{F}, \mathbf{T}\}$
- formula encodes constraints
- solution: satisfying assignment of values to variables
- SAT algorithms for this problem: $\rightsquigarrow$ DPLL

Propositional Logic: Description of State Spaces

propositional variables for missionaries and cannibals problem:

```
two-missionaries-are-on-left-shore
one-cannibal-is-on-left-shore
boat-is-on-left-shore
...
```

- problem description for general problem solvers
- states represented as truth values of atomic propositions

Propositional Logic: Intuition

propositions: atomic statements over the world that cannot be divided further

Propositions with logical connectives like "and", "or" and "not" form the propositional formulas.

Syntax

Syntax

Σ alphabet of propositions
(e.g., $\{P, Q, R, \dots\}$ or $\{X_1, X_2, X_3, \dots\}$).

Definition (propositional formula)

- $\top$ and $\bot$ are formulas.
- Every proposition in Σ is an (atomic) formula.
- If φ is a formula, then $\neg\varphi$ is a formula (negation).
- If φ and ψ are formulas, then so are
  - $(\varphi \land \psi)$ (conjunction)
  - $(\varphi \lor \psi)$ (disjunction)
  - $(\varphi \to \psi)$ (implication)

binding strength: $(\neg) > (\land) > (\lor) > (\to)$
(may omit redundant parentheses)

Semantics

Semantics

A formula is true or false,
depending on the interpretation of the propositions.

Semantics: Intuition

- A proposition p is either true or false.
  The truth value of p is determined by an interpretation.
- The truth value of a formula follows from
  the truth values of the propositions.

Example

$\varphi = (P \vee Q) \wedge R$

- If P and Q are false, then φ is false
  (independent of the truth value of R).
- If P and R are true, then φ is true
  (independent of the truth value of Q).

Semantics: Formally

- defined over interpretation $I : \Sigma \to \{\mathbf{T}, \mathbf{F}\}$
- interpretation I: assignment of propositions in Σ
- When is a formula φ true under interpretation I?
  symbolically: When does $I \models \varphi$ hold?

Semantics: Formally

Definition ($I \models \varphi$)

- $I \models \top$ and $I \not\models \bot$
- $I \models P$ iff $I(P) = \mathbf{T}$ for $P \in \Sigma$
- $I \models \neg\varphi$ iff $I \not\models \varphi$
- $I \models (\varphi \wedge \psi)$ iff $I \models \varphi$ and $I \models \psi$
- $I \models (\varphi \vee \psi)$ iff $I \models \varphi$ or $I \models \psi$
- $I \models (\varphi \rightarrow \psi)$ iff $I \not\models \varphi$ or $I \models \psi$
- $I \models \Phi$ for a set of formulas Φ iff $I \models \varphi$ for all $\varphi \in \Phi$

Examples

Example (Interpretation I)

$I = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{F}, S \mapsto \mathbf{F}\}$

Which formulas are true under I?

- $\varphi_1 = \neg(P \wedge Q) \wedge (R \wedge \neg S)$. Does $I \models \varphi_1$ hold?
- $\varphi_2 = (P \wedge Q) \wedge \neg(R \wedge \neg S)$. Does $I \models \varphi_2$ hold?
- $\varphi_3 = (R \rightarrow P)$. Does $I \models \varphi_3$ hold?

Terminology

Definition (model)

An interpretation I is called a model of φ if $I \models \varphi$.

Definition (satisfiable etc.)

A formula φ is called

- satisfiable if there is an interpretation I such that $I \models \varphi$.
- unsatisfiable if φ is not satisfiable.
- falsifiable if there is an interpretation I such that $I \not\models \varphi$.
- valid (= a tautology) if $I \models \varphi$ for all interpretations I.

Terminology

Definition (logical equivalence)

Formulas φ and ψ are called logically equivalent ($\varphi \equiv \psi$) if for all interpretations I: $I \models \varphi$ iff $I \models \psi$.

Truth Tables

Truth Tables

How to determine automatically if a given formula is (un)satisfiable, falsifiable, valid?

$\rightsquigarrow$ simple method: truth tables

example: Is $\varphi = ((P \lor H) \land \neg H) \rightarrow P$ valid?

P	H	$P \lor H$	$((P \lor H) \land \neg H)$	$((P \lor H) \land \neg H) \rightarrow P$
F	**F**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**
T	**T**	**T**	**F**	**T**

$I \models \varphi$ for all interpretations I $\rightsquigarrow$ φ is valid.

- satisfiability, falsifiability, unsatisfiability?

Normal Forms

Normal Forms: Terminology

Definition (literal)

If $P \in \Sigma$, then the formulas P and $\neg P$ are called literals.

P is called positive literal, $\neg P$ is called negative literal.

The complementary literal to P is $\neg P$ and vice versa.
For a literal ℓ, the complementary literal to ℓ is denoted with $\bar{\ell}$.

Normal Forms: Terminology

Definition (clause)

A disjunction of 0 or more literals is called a clause.
The empty clause $\bot$ is also written as $\square$.
Clauses consisting of only one literal are called unit clauses.

Definition (monomial)

A conjunction of 0 or more literals is called a monomial.

Normal Forms

Definition (normal forms)

A formula φ is in conjunctive normal form (CNF, clause form) if φ is a conjunction of 0 or more clauses:

$$\varphi = \bigwedge_{i=1}^{n} \left(\bigvee_{j=1}^{m_i} \ell_{i,j} \right)$$

A formula φ is in disjunctive normal form (DNF) if φ is a disjunction of 0 or more monomials:

$$\varphi = \bigvee_{i=1}^{n} \left(\bigwedge_{j=1}^{m_i} \ell_{i,j} \right)$$

Normal Forms

For every propositional formula, there exists
a logically equivalent propositional formula in CNF and in DNF.

Conversion to CNF

important rules for conversion to CNF:

- $(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)$ (($\rightarrow$)-elimination)
- $\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$ (De Morgan)
- $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$ (De Morgan)
- $\neg\neg\varphi \equiv \varphi$ (double negation)
- $((\varphi \wedge \psi) \vee \eta) \equiv ((\varphi \vee \eta) \wedge (\psi \vee \eta))$ (distributivity)

There are formulas φ for which every logically equivalent formula
in CNF and DNF is exponentially longer than φ.

Summary

Summary (1)

- Propositional logic forms the basis for a general representation of problems and knowledge.
- Propositions (atomic formulas) are statements over the world which cannot be divided further.
- Propositional formulas combine atomic formulas with $\neg$, $\wedge$, $\vee$, $\rightarrow$ to more complex statements.
- Interpretations determine which atomic formulas are true and which ones are false.

Summary (2)

- important terminology:
  - model
  - satisfiable, unsatisfiable, falsifiable, valid
  - logically equivalent
- different kinds of formulas:
  - atomic formulas and literals
  - clauses and monomials
  - conjunctive normal form and disjunctive normal form